

- ① $\text{int} \left\{ p \in C[a,b] \mid p(x) = \sum_{j=0}^n a_j x^j \right\} = \emptyset$
- ② $\{ p(e^x) \in C[a,b] \mid p \text{ polynomial} \}$ is dense in $C[a,b]$
- ③ $\{ e^{p(x)} \in C[a,b] \mid p \text{ poly.} \}$ is not dense
- ④ $\{ p(\cos(x)) \mid p \text{ poly.} \}$ is dense in $C[0, \pi/2]$, but not dense in $C[-1, 1]$.

let (X, d) be a metric space.

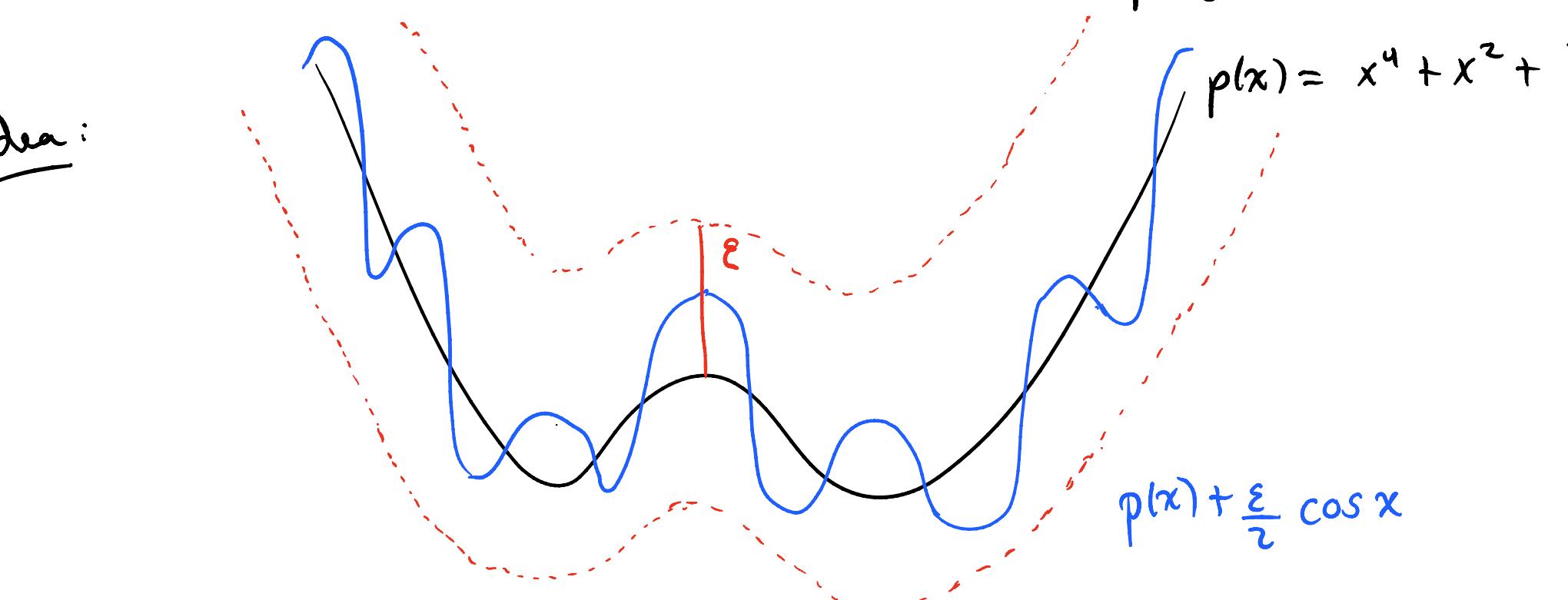
Defn: $S \subseteq X$ is dense $\Leftrightarrow \forall x \in X, \forall \varepsilon > 0, B(\varepsilon, x) \cap S \neq \emptyset$.

PROBLEM 1 Given any polynomial $p(x)$ and any $\varepsilon > 0$, it suffices to show

$$B(\varepsilon, p) = \left\{ f \in C[a,b] \mid \sup_{x \in [a,b]} |f(x) - p(x)| < \varepsilon \right\}$$

Contains functions which are not polynomials.

Idea:



Take p , and add a "small" piece so that it is no longer polynomial.

Define $f(x) = p(x) + \frac{\varepsilon}{2} \cos x$, which is not polynomial.

$$\sup_{x \in [a,b]} |f(x) - p(x)| = \sup_{x \in [a,b]} \frac{\varepsilon}{2} |\cos x| \leq \frac{\varepsilon}{2} < \varepsilon$$

So $f \in B(\varepsilon, p)$ This completes the proof that

So $t \in B(\varepsilon, p)$. This completes the proof that
 $\text{int}\{\text{polynomials}\} = \emptyset$. Notice that the polynomials are
dense in $C[a,b]$ by Weierstrass' theorem, but have empty
interior (think $\mathbb{Q} \subseteq \mathbb{R}$!).

PROBLEM 2 The e^x is kind of a red herring, as we will see.

Given any $\varepsilon > 0$ and $f \in C[a,b]$, we want to find $p(e^x)$ s.t.

$$\sup_{x \in [a,b]} |f(x) - p(e^x)| < \varepsilon \quad (*)$$

Set $t = e^x$, then

① $f(\log t)$ is continuous on $[e^a, e^b]$

② $|f(x) - p(e^x)| = |f(\log t) - p(t)|$

where $p(t)$ is a polynomial in t .

By density of polynomials, we can find $p(t) = \sum_{j=1}^n a_j t^j$ s.t.

$$\sup_{t \in [e^a, e^b]} |f(\log t) - p(t)| < \varepsilon$$

Then $p(e^x)$ satisfies condition (*).

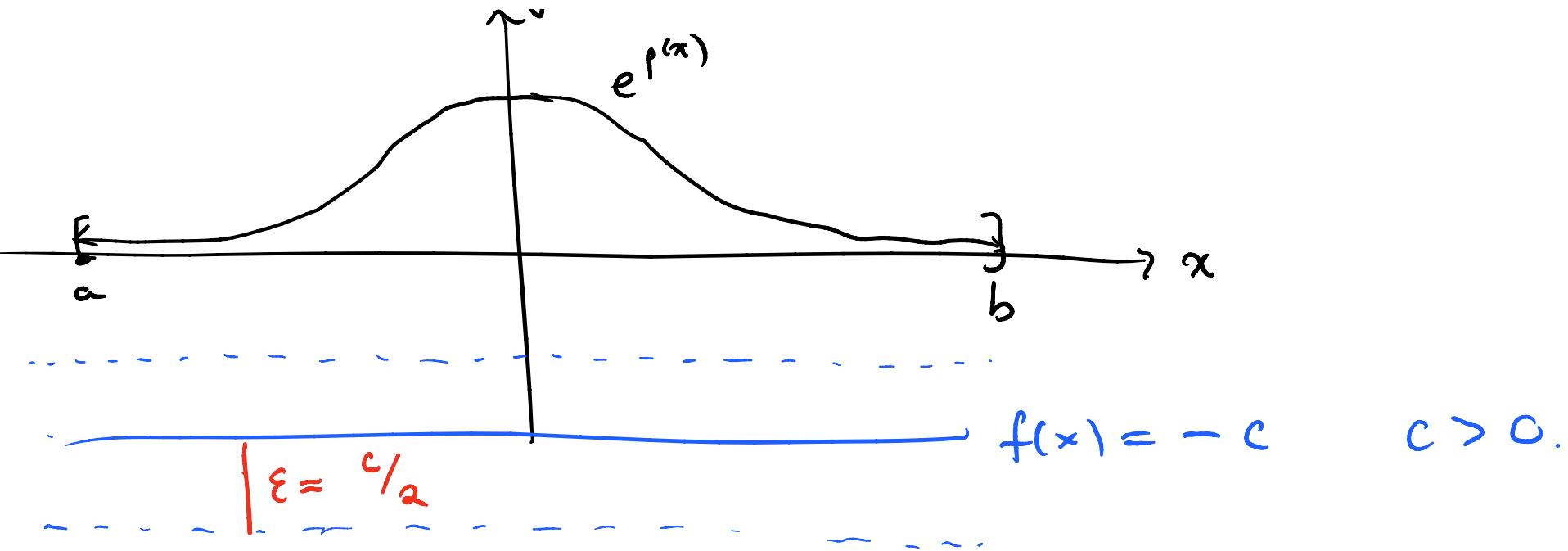
PROBLEM 3 We must prove that $\exists f \in C[a, b]$ and $\exists \varepsilon > 0$,

$$B(\varepsilon, f) \cap \{e^{p(x)} \mid p \text{ poly}\} = \emptyset$$

Notice that $e^y > 0 \quad \forall y \in \mathbb{R}$, so for $y = p(x)$ we also have

$$e^{p(x)} > 0 \quad \forall x.$$

1neu



Exercise: Complete the proof that $B(f, \epsilon) \cap \{e^{p(x)} | p \text{ poly}\} = \emptyset$

PROBLEM 4: Showing that $\{p(\cos x) | p \text{ poly}\}$ is dense in $C[0, \pi/2]$ is the same as problem 2, setting $x = \arccos t$.

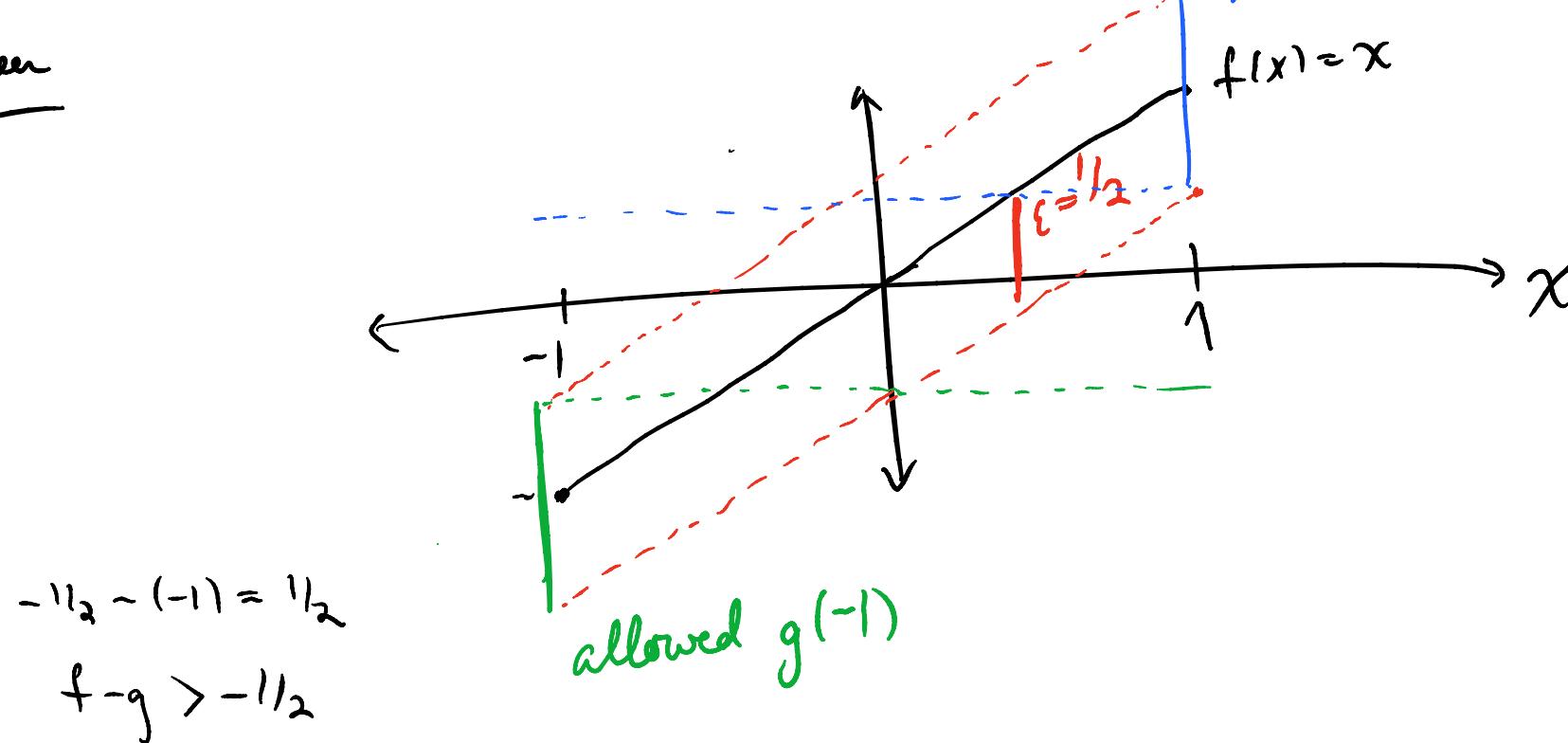
So why is this family not dense in $C[-1, 1]$? Need to show $\exists f \in C[-1, 1]$ and $\epsilon > 0$ s.t. $B(\epsilon, f) \cap \{p(\cos x) | p \text{ poly}\} = \emptyset$

Notice that any function of the form $p(\cos x)$ is even, so

we just need to find a function f and an ε so that

if $g \in B(f, \varepsilon)$ then $g(x) \neq g(-x)$

Idea



Set $f(x) := x$, and let $\varepsilon = 1/2$.

Claim: If $g \in B(f, \varepsilon)$ then $\exists x \in [-1, 1]$ s.t. $g(x) \neq g(-x)$

Proof: Consider $x=1$ Then since $f(1)-g(1) < 1/2$ and $f(1)-g(1) > -1/2$,

$$g(1) = f(1) - (f(1) - g(1)) > f(1) - \frac{1}{2} = \frac{1}{2}$$

$$g(-1) = f(-1) - (f(-1) - g(-1)) < f(-1) + \frac{1}{2} = -\frac{1}{2}$$

then since $g(1) > \frac{1}{2}$ and $g(-1) < -\frac{1}{2}$ we could never have $g(1) = g(-1)$. \square

This has shown that if $\forall x \quad g(x) = g(-x)$ then $g \notin B(f, \varepsilon)$.

which completes the proof that $\{p(\cos x) \mid p \text{ poly.}\}$ is not dense.